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PROGRESS REPORT.

1. Prof. Davi Dayal, B.A., of Lahore having expressed a desire to retire from the Committee, the Committee are pleased to appoint, in his place, Prof. Gopal singh Chowla, M. A., Government College. Lahore, for the remaining part of this year.

2. According to Art. VIII (c) of our Constitution the General Body having voted unanimously on the proposal for the reconstitution of the Committee for the next year, the following gentlemen will form the Committee for the year 1912 :

1.	Prof. R. N. Apte, M.A., LL.B., F.R. A.S	President
2.	The Rev. C. Pollard, M.A.	Treasurer
3.	Principal R. P. Paranjpye, M.A., B Sc. ...	Librarian.
4.	Mr. Balakram, M.A., I.C.S.	Members.
5.	Mr. K. Balaram Iyer, B.A.	
6.	Prof. Gopal Singh Chowla, M.A. ...	
7.	,, B. Hanumanta Rau, B.A. ...	
8.	Mr. D. D. Kapadia, M.A., B.Sc. ...	
9.	Mr. M. P. Khareghat, I.C.S., (retired.) ...	
10.	Mr. M. T. Naraniengar, M.A.	
11.	Mr. V. Ramaswami Aiyar, M A. ...	
12.	Mr. R. Swaminatha Iyer, B.A.... ...	

3. The following gentlemen have been elected members of the Society :

1. Mr. A. A. Krishnasawmi Aiyangar, B.A., Clerk, Accountant General's office, Madras (*at concessional rate*).
2. Mr. S. B. Butani, M.A., B.Sc., Assistant Professor of Physics and Mathematics, Agricultural College, Poona.
3. Mr. P. M. Advani, B.A., B.Sc., Lecturer in Physics and Chemistry, Dayaram Jethmal Sind College, Karachi.
4. Mr. D. B. Advani, M.A., LL. B., Pleader, Hyderabad, (Sind).
5. Mr. Keshavlal D. Shah, Senior Mathematical Master, Proprietory High School, Ahmedabad.

4. The following new books have been purchased for the Library:—

(1) *Rara Arithmetica*: by Prof. D. E. Smith, 21*s*., Published by Messrs. Ginn & Co., Boston;

(2) *Diophantus of Alexandria*—a study in the History of Greek Algebra: by Sir Thomas, L. Heath Sc.D., 12/6*d*. University Press, Cambridge.

5. The following books have been received as presents from the publishers:—

1. *Theory of Determinants.*—in the Historical Order of Development, Vol. II; by Dr. Thomas Muir, LL D. 17 *s*, *net*. Macmillan & Co.
2. *Projective Geometry*—for use in Colleges and Schools; by W. P. Milne, M.A., B.Sc., 2*s*. 6*d*. Macmillan & Co.
3. Bombay University Calendar for 1911-12. Vols. I & II.
4. Punjab University Calendar for 1911.

POONA,
30th Nov. 1911.

D. D. KAPADIA,
Hon. Joint Sec.

The General Equation in the Algebra of Logic.

By Balakram I.C.S.

§ 1. A set of propositions regarding n logical classes may be represented by a set of equations containing n unknowns. In the analytic processes of deduction (represented by the solution of the equations) we replace the n (possibly over-lapping) classes by a number (in general greater than n) of mutually exclusive classes the existence of which is implied by the original n classes; though in the simpler processes (where a few logical steps may be safely omitted) the amplification need not be explicit, and in expressing the final result the classes may be regrouped into the original n (or any other) classes. For instance if $n=2$, the two original classes x, y, may be amplified into four classes:

xy (both x and y);
$x\bar{y}$ (x but not y);
$\bar{x}y$ (y but not x);
$\bar{x}\bar{y}$ (neither x nor y);

connected by the identical relation,

$$xy+\bar{x}y+x\bar{y}+\bar{x}\bar{y}=\text{the universe}=1.$$

We may (if convenient) divide x, y into only three classes (xy, $x\bar{y}$, $\bar{x}$) which also, taken together, exhaust the universe. For the n original mutually exclusive classes that we have to consider, may thus number 2^n at the most.

§ 2. By means of the propositions

(1) that the assertion $a=b$ is equivalent to two denials $a\bar{b}=0$, $\bar{a}b=0$; and

(2) that these denials are together equivalent to the single denial $a\bar{b}+\bar{a}b=0$,

a given set of propositions may (after the classes have been amplified as explained above) be represented by the single equation

$$a_1x_1+a_2x_2+\; \cdot\;\cdot\;\cdot\;\cdot\;\cdot \;+a_nx_n=0$$

where the a's are constants (arising out of the data), and the x's (the unknowns) are connected by the relations $x_rx_s=0$ ($r \neq s$) for all r and s.

There is no loss of generality in assuming $\Sigma x=1$. For if this relation do not hold, we introduce a new class $x_{n+1}=\bar{x}_1\bar{x}_2\ldots\ldots\bar{x}_n$, and associate with it a zero coefficient. Our equation is now

$$a_1x_1+a_2x_2+\ldots\ldots+a_nx_n+ox_{n+1}=0$$

which is of the required form.

§ 3. The standard type of equation may thus be taken to be

$$\sum_1^n a_r x_r = 0,$$

where $x_r x_s = 0$ $(r \neq s)$ for all r and s, and

$$\sum_1^n x_r = 1$$

The number of independent unknowns is $n-1$.

§ 4. The method of solution is that stated in Whitehead's Universal Algebra § 36 to be due to Johnson, some of the details being however different. New and simple formulae are obtained applicable to the general equation involving unknowns limited as above, and also to the general equation in which the condition $\sum_1^n x_r = 1$ is not satisfied.

§ 5. Johnson begins by solving the simultaneous equations $x < a$, $y < b$, $xy = 0$, $x + y = a + b$ [$x < a$ stands for "all x is a," and is therefore equivalent to $x = ax$ or $\bar{a}\, x = 0$. We may also write $a > x$ for $x < a$.]

$$x + y = a + b = a\bar{b} + ab + \bar{a}b.$$

Thus three mutually exclusive classes $a\bar{b}$, ab, $\bar{a}b$, are to be divided into two mutually exclusive classes x and y, such that ab does not fall in x, and $a\bar{b}$ does not fall in y. The solution obviously is

$$x = a\bar{b} + \lambda . ab$$

$$y = \bar{a}b + \bar{\lambda} . ab$$

where λ is arbitrary.

This may be written

$$x = a(\bar{b} + \lambda b) = a(\bar{b} + \lambda) ;$$

and $$y = b(\bar{a} + \bar{\lambda} a) = b(\bar{a} + \bar{\lambda}).$$

We generalize this proposition as follows :—

If $x < a$, $y < b$, $xy = 0$, and $x + y = c(a + b)$, then $x = ca(\bar{b} + \lambda)$, and $y = cb(\bar{a} + \bar{\lambda})$.

For $x + y = c(a + b)$, $\therefore$ $x + y < c$.

$\therefore$ $x < c$. Also $x < a$; $\therefore$ $x < ac$. Similarly $y < bc$.

Thus $x < ac$, $y < bc$, $xy = 0$, $x + y = ac + bc$.

$\therefore$ $x = ac(\bar{b} + \bar{c} + \lambda) = ac(\bar{b} + \lambda)$ and $y = bc(\bar{a} + \bar{c} + \bar{\lambda}) = bc(\bar{a} + \bar{\lambda})$.

§6. E $(=a_1a_2...a_n)$ is called the eliminant of the expression $\sum_1^n a_rx_r$.

¶ It is well-known that if $\sum_1^n a_rx_r = 0$, and $\sum_1^n x_r = 1$, then $E=0$, and that this is the only relation (not involving the unknowns) derivable from the equation. That $E=0$, may be proved as follows :—

$$E = E\sum_1^n x_r = \sum_1^n Ex_r = \sum_1^n (a_1a_2...a_rx_r...a_n) = 0.$$

Taking the supplement

$$\overline{E}\sum_1^n a_r = 1.$$

§. 7. From $\sum_1^n a_rx_r = 0$, we get

$$x_r < \overline{a}_r \text{ for all } r,$$

and
$$\sum_1^n a_r = 1 = \sum_1^n \overline{x}_r$$

Thus
$$x_1 < \overline{a}_1 ; \quad \sum_2^n x_r < \sum_2^n \overline{a}_r ;$$

$$x_1 + \sum_2^n \overline{x}_r = a_1 + \sum a_r$$

and
$$x_1 . \sum_2^n x_r = 0.$$

Therefore by the lemma,

$$x_1 = \overline{a}_1 (\lambda_1 + a_2a_3 .... a_n)$$

and
$$\sum_2^n x_2 = (a_1 + \overline{\lambda}_1) \sum_2^n \overline{a}_r.$$

We have now

$$\overline{x}_2 < a_2, \quad \sum_0^n x_r < \sum_3^n a_r$$

$$x_2 + \sum_3^n x_r = [\overline{a}_2 + \sum_3^n \overline{a}_r](a_1 + \overline{\lambda}_1)$$

and
$$x_2 \sum_3^n x_r = 0$$

Therefore by the Lemma,

$$x_2=(a_1+\bar{\lambda}_1)\ a_2-(\lambda_2+a_3a_4\ldots\ldots\ldots a_n)$$

$$\sum_3^n x_3=(a_1+\bar{\lambda}_1)(a_2+\bar{\lambda}_2)\sum_3^n a_r.$$

Continuing the process, we get

$$x_r=(a_1+\bar{\lambda}_1)(a_2+\bar{\lambda}_2)\ \ldots\ldots.(a_{r-1}+\bar{\lambda}_{-1})\ a_r\ (\lambda_r+a_{r+1}\ldots a_n),$$

for all values of r. The number of arbitrary constants λ is $n-1$, equal to the number of independent unknowns.

By repeated applications of the formula

$$k+ab=(k+a)\ (k+b)$$

we may write

$$x_r=(a_1+\bar{\lambda}_1)(a_2+\bar{\lambda}_2)\ .\ \bar{a}_r\ (a_{r+1}+\lambda_r)(a_{r+1}+\lambda_r)\ldots(a_n+\lambda_r).$$

Taking the supplement

$$\bar{x}_r=\lambda_1\bar{a}_1++\lambda_2\bar{a}_2+\ldots\ldots\ldots+a_r+\bar{\lambda}_r\ (\bar{a}_{r+1}+\ldots\ldots+\bar{a}_n)$$

These solutions are easy to remember.

Cor. 1. If $a_r=1$, $\bar{a}_r=0$, and $x_r=0$. Also $a_r+\lambda=1$. The terms involving a_r and $\bar{a}_r$ disappear in the expressions for the various x's. In solving the equation we may consequently ignore terms like x_r which have a coefficient unity.

Cor. 2. If $\bar{a}_ra_s=0$, the factor $a_s+\lambda$ may be replaced by λ; for $\bar{a}_r\ (a_s+\lambda)=\bar{a}_r\lambda$.

Cor. 3. If $\bar{a}_ra_s=o$ and $s>r$, the factor $(\lambda_r+a_{r+1}\ a_{r+2}\ldots\ldots a_n)$ may be replaced by λ. In such a case.

$$x_r=(a_1+\bar{\lambda}_1)\ldots\ldots\bar{a}_r.\ \lambda_r$$

Cor. 4. If $\bar{a}_r\bar{a}_s=o$, the factor $a_s+\lambda$ may be omitted in the expression for x_r. For $\bar{a}_r\ (a_s+\lambda)=\bar{a}_r\ (a_s+\bar{\lambda}a_s)=\bar{a}_ra_s=\bar{a}_r$. The number of arbitrary constants in the expressions for x_r and x_s is reduced by one in virtue of the relation $\bar{a}_r\bar{a}_s=o$.

Cor. 5. If $\bar{a}_r\bar{a}_s=o$ $(r\neq s)$ for all r and s, the arbitrary constants disappear, and the values of the unknowns become definite,

i.e., $x_r=\bar{\bar{a}}_r.$

We see that the indeterminate we sometimes meet with in results arrived at by deduction, is solely due to the overlapping of classes a in which the unknown, the x's, are asserted to exist one by one.

§ 8. It is sometimes convenient and possible to reduce provisionally the number of unknowns. For instance: if $x_{n-1}=x_{n-1}$, we replace x_1+x_2 by a single unknown ξ_{n-1}. Our new equation

$$a_1x_1+a_3x_2+\ldots\ldots+a_{n-3}\ x_{n-2}+a_{n-1}\ \xi_{n-1}=0$$

involves only $n-1$ unknowns, and the groups of solutions will contain $n-2$ arbitrary constants. The missing constant reappears when we seek to obtain the values of x_{n-1} and x_n from that of ξ_{n-1}; for

$$x_{n-1} < \xi_{n-1} \qquad x_{n-1} < \xi_{n-1} \qquad x_{n-1} + x_n = \xi_{n-1} \qquad x_{n-1}\, x_n = 0$$

$$\therefore \qquad x_{n-1} = \overline{\xi_{n-1}}\,(\xi_{n-1} + \lambda) = \xi_{n-1}\lambda$$

and

$$x_n = \overline{\xi_{n-1}}\lambda$$

§ 9. The order in which the x's are evaluated is arbitrary. The formal expression for x_r will depend upon the particular order in which the evaluation is carried out; hence there will be formally $n\,!$ sets of solutions. The conditions of the problem will suggest the most convenient order; having written down the solutions we may (if desired) add two or more of the x's to reproduce the original over-lapping classes.

§ 10. We are now in a position to solve the equation $\sum_1^n a_r x_r$, where

$\sum_1^n x_r \neq 1$.

Introducing the class $x_{n+1} = \bar{x}_1\, \bar{x}_2 \ldots \bar{x}_n$, and writing a_{n+1} for 0, our equation becomes

$$\sum_1^{n+1} a_r\, x_r = 0 \qquad \sum_1^{n+1} x_r = 1.$$

The solution is

$$x_r = (a_1 + \bar{\lambda}_1)\,(a_2 + \bar{\lambda}_2) \ldots a_r\,(\lambda_r + a_{r+1} a_{r+2} \ldots a_{n+1})$$

Substituting for a_{n+1} its value 0, we get

$$x_r = (a_1 + \bar{\lambda}_1)\,(a_2 + \bar{\lambda}) \ldots \bar{a}_r\,.\,\lambda_r, \text{ for } r = 1, 2 \ldots n,$$

which gives one set of solutions.

Another simple set is obtained by evaluating x_{n+1} first. Writing μ for $\bar{\lambda}_{n+1}$

$$x_{n+1} = \bar{a}_{n+1}\,(\mu + a_1\, a_2 \ldots a_n)$$

$$= \bar{\mu} + a_1\, a_2 \ldots a_n$$

and

$$x_r = \bar{\mu}\overline{(a_1 + \lambda_1)}\,(a_2 + \bar{\lambda}_2) \ldots a_r\,(a_{r+1} + \lambda_r) \ldots (a_n + \lambda_r)$$

for $r = 1, 2 \ldots n$.

The solution contains n arbitrary constants.

§ 11. As an illustration of the method, we solve the set of equations

$$yz = a,\; zx = b,\; xy = c,$$

discussed by Boole, Venn, and Whitehead.

The equations are equivalent to

$$\bar{a}yz+a(\bar{y}+\bar{z})+\bar{b}xz+b(\bar{x}+\bar{z})+\bar{c}xy+c(\bar{x}+\bar{y})=0,$$

or (amplifying), to

$$(\bar{a}+\bar{b}+\bar{c})xyz+(\bar{a}+b+c)\bar{x}yz+(a+\bar{b}+c)x\bar{y}z$$
$$+(a+b+\bar{c})xy\bar{z}+(a+b+c)(\bar{x}\bar{y}z+\bar{x}y\bar{z}+x\bar{y}\bar{z}+\bar{x}\bar{y}\bar{z})=0,$$

the four classes in the last bracket being provisionally treated as one.

The eliminant is

$$(\bar{a}+\bar{b}+\bar{c})(\bar{a}+b+c)(a+\bar{b}+c)(a+b+\bar{c})(a+b+c)$$
$$=(\bar{a}+\bar{b}+\bar{c})(b+c)(c+a)(a+b)$$
$$=(\bar{a}+\bar{b}+\bar{c})(ab+bc+ca)$$
$$=\bar{a}bc+a\bar{b}c+ab\bar{c}=0$$

The constants of the equation are such that $\bar{a}_r\bar{a}_s=0$ for all r and s.

Therefore by § 7, Cor. 5,

$$xyz=abc,\ x\bar{y}z=a\bar{b}c,\ xy\bar{z}=ab\bar{c},\ \bar{x}yz=\bar{a}bc\ ;$$

and $$\bar{x}\bar{y}z+\bar{x}y\bar{z}+x\bar{y}\bar{z}+\bar{x}\bar{y}\bar{z}=\bar{a}\bar{b}\bar{c}$$

From the last of these, we have at once

$$\bar{x}\bar{y}z=\lambda.\bar{a}\bar{b}\bar{c}$$
$$\bar{x}y\bar{z}=\bar{\lambda}\mu.\bar{a}\bar{b}\bar{c}$$
$$x\bar{y}\bar{z}=\bar{\lambda}\bar{\mu}\nu.\bar{a}\bar{b}\bar{c}$$

and $$\bar{x}\bar{y}\bar{z}=\bar{\lambda}\bar{\mu}\bar{\nu}.\bar{a}\bar{b}\bar{c}$$

To find x:

$$x=x(yz+y\bar{z}+\bar{y}z+\bar{y}\bar{z})$$
$$=abc+a\bar{b}c+ab\bar{c}+\bar{\lambda}\bar{\mu}\nu.\bar{a}\bar{b}\bar{c}$$

Using the relations given by the eliminant, and simplifying

$$x=b+c+\bar{\lambda}\bar{\mu}\nu.\bar{a}.$$

Similarly for y and z.

The eliminant may be solved for c :—

$$(\bar{a}b+a\bar{b})c+ab\bar{c}=0,$$

$$\therefore\quad c=(ab+\bar{a}\bar{b})(k+ab).$$
$$=ab+k\bar{a}\bar{b}.$$

and $$\bar{c}=\bar{a}b+a\bar{b}+\bar{k}\bar{a}\bar{b}.$$

Some Properties of Bernoulli's Numbers.

By S. Ramanujan.

1. Let the well known expansion of $x.\cot x$ (*vide*: Edward's *Diff Cal.*, § 149) be written in the form

$$x\cot x = 1 - \frac{B_2}{\underline{|2}}(2x)^2 - \frac{B_4}{\underline{|4}}(2x)^4 - \frac{B_6}{\underline{|6}}(2x)^6 - \&c. \;\dots\dots\dots\dots\dots (1)$$

from which we infer that B_0 may be supposed to be -1.

Now,
$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \&c.}{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \&c.}$$

$$= \frac{\sin 2x}{1-\cos 2x} = \frac{\frac{2x}{\underline{|1}} - \frac{(2x)^3}{\underline{|3}} + \frac{(2x)^5}{\underline{|5}} - \&c.}{\frac{(2x)^2}{\underline{|2}} - \frac{(2x)^4}{\underline{|4}} + \frac{(2x)^6}{\underline{|6}} - \&c.}$$

$$= \frac{1+\cos 2x}{\sin 2x} = \frac{2 - \frac{(2x)^2}{\underline{|2}} + \frac{(2x)^4}{\underline{|4}} - \frac{(2x)^6}{\underline{|6}} + \&c.}{2x - \frac{(2x)^3}{\underline{|3}} + \frac{(2x)^5}{\underline{|5}} - \frac{(2x)^7}{\underline{|7}} + \&c.}$$

Multiplying both sides in each of the above three relations by the denr. of R. H. S. and equating the coeffts. of x^n on both sides, we can write the results thus :—

$$c_1\frac{B_{n-1}}{2} - c_3\frac{B_{n-3}}{2^3} + c_5\frac{B_{n-5}}{2^5} - \dots + \frac{(-1)^{\frac{n-1}{2}}}{2^n}B_0 + \frac{n}{2^n}(-1)^{\frac{n-1}{2}} = 0 \;\dots\quad (2)$$

where n is any odd integer.

$$c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \dots + (-1)^{\frac{n-2}{2}}B_0 + \frac{n}{2}(-1)^{\frac{n-2}{2}} = 0 \quad\dots\quad (3)$$

where n is any even integer.

$$c_1 B_{n-1} - c_3 B_{n-3} + c_5 B_{n-5} - \dots + (-1)^{\frac{n-1}{2}}B_0 + \frac{n}{2}(-1)^{\frac{n-1}{2}} = 0 \quad\dots\quad (4)$$

where n is any odd integer greater than unity.

From any one of (2), (3), (4) we can calculate the B's. But as n becomes greater and greater the calculation will get tedious. So, we shall try to find simpler methods.

2. We know

$$(x\cot x)^2 = -x^2\left(1 + \frac{d\cot x}{dx}\right)$$

Using (1) and equating the co-efficients of x^n on both sides, and simplifying we have

$$\frac{n+1}{2}B_n = c_2\, B_2\, B_{n-2} + c_4\, B_4\, B_{n-4} + c_6\, B_6\, B_{n-6} + \&c.,$$

the last term being $C_{\frac{n-1}{2}} B_{\frac{n-1}{2}} B_{\frac{n+1}{2}}$ or $c_{\frac{n}{2}}\left(B_{\frac{n}{2}}\right)^2$ according as $n/2$ is odd or even ... ... (5)

A similar result can be obtained by equating the co-efficients of x^n in the identity

$$\frac{d \tan x}{dx} = 1 + \tan^2 x$$

3. Again

$$-\frac{x}{2}\left(\cot\frac{x}{2} + \coth\frac{x}{2}\right) = -\frac{x}{2}\left(\cot\frac{x}{2} + i\cot\frac{ix}{2}\right)$$

$$= 2\left\{B_0 + B_4\frac{x^4}{\lfloor 4} + B_8\frac{x^8}{\lfloor 8} + \ldots\ldots\right\}$$

by using (1). The expression may also be written

$$-\frac{x}{2}\,\frac{\left(\cos\frac{x}{2}\sin\frac{ix}{2} + i\sin\frac{x}{2}\cos\frac{ix}{2}\right)}{\sin\frac{x}{2}\sin\frac{ix}{2}} = -\frac{x}{2}\,\frac{(1+i)\sin\frac{1}{2}x(1+i) - (1-i)\sin\frac{1}{2}x(1-i)}{\cos\frac{1}{2}x(1-i) - \cos\frac{1}{2}x(1+i)}$$

$$= -x.\ \frac{\left\{\frac{x}{\lfloor 1} - \frac{x^5}{2^2\lfloor 5} + 2^4\frac{x^9}{\lfloor 9} - \ldots\right\}}{\left\{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2\lfloor 6} + \frac{x^{10}}{2^4\lfloor 10} - \ldots\right\}}$$

by expanding the numerator and the denominator, and simplifying by De Moivre's theorem.

Hence

$$2\left(B_0 + B_4\frac{x^4}{\lfloor 4} + B_8\frac{x^8}{\lfloor 8} + \ldots\right) = -x.\ \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^2\lfloor 5} + \ldots}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2\lfloor 6} + \ldots} \quad \ldots (6)$$

Similarly

$$-\frac{x}{2}\left(\cot\frac{x}{2} - \coth\frac{x}{2}\right) = 2\left(B_2\frac{x^2}{\lfloor 2} + B_6\frac{x^6}{\lfloor 6} + B_{10}\frac{x^{10}}{\lfloor 10} + \&c.\right)$$

$$= x.\ \frac{\frac{1-i}{2}\sin\frac{x(1+i)}{2} - \frac{1+i}{2}\sin\frac{x(1-i)}{2}}{\cos\frac{x(1+i)}{2} - \cos\frac{x(1-i)}{2}}$$

$$= x.\frac{\frac{x^3}{2\,\underline{|3}} - \frac{x^7}{2^3\,\underline{|7}} + \frac{x^{11}}{2^5\,\underline{|11}} - }{\frac{x^2}{\underline{|2}} - \frac{x^6}{2^2\,\underline{|6}} + \frac{x^{10}}{2^4\,\underline{|10}} - } \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Proceeding as in § 1 we have if n is an even integer greater than 2

$$c_2\frac{B_{n-2}}{2} - c_6\frac{B_{n-6}}{\underline{|2^3}} + c_{10}\frac{B_{n-10}}{\underline{|2^5}} - \ldots \frac{n}{2^{\frac{n+2}{2}}}(-1)^{\frac{n}{4}} \text{ or } \frac{n}{2^{\frac{n+2}{2}}}(-1)^{\frac{n-2}{4}} = 0 \quad (8)$$

according as n or $n-2$ is a multiple of 4.

Analagous results can be obtained from $\tan x/2 \pm \tanh x/2$.

In (2), (3) and (4) there are $n/2$ terms, while in (5) and (8) there are $n/4$ or $(n-2)/4$ terms. Thus B_n can be found from only half of the previous B's.

4. A still simpler method can be deduced from the following identities.

If 1, ω, ω^2 be the three cube roots of unity, then

$$4 \sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2),$$

as may easily be verified.

By logarithmic differentiation, we have

$$\cot x + \omega \cot x\omega + \omega^2 \cot x\omega^2 = 2\frac{\cos 2x + \omega\cos 2x\omega + \omega^2\cos 2x\omega^2}{\sin 2x + \sin 2x\omega + \sin 2x\omega^2}$$

Writing $\frac{x}{2}$ for x

$$-\frac{x}{2}\left(\cot\frac{x}{2} + \omega\cot\frac{x\omega}{2} + \omega^2\cot\frac{x\omega^2}{2}\right) = -x.\frac{\cos x + \omega\cos x\omega + \omega^2\cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2},$$

and proceeding as in § 3 we get

$$3\left(B_0 + B_6\frac{x^6}{\underline{|6}} + B_{12}\frac{x^{12}}{\underline{|12}} + \&c.\right) = -x.\frac{\frac{x^2}{\underline{|2}} - \frac{x^8}{\underline{|8}} + \frac{x^{14}}{\underline{|14}} - \&c.}{\frac{x^3}{\underline{|3}} - \frac{x^9}{\underline{|9}} + \frac{x^{15}}{\underline{|15}} - \&c.} \quad \ldots \quad (9)$$

Again $\cot\frac{x\omega}{2} - \cot\frac{x\omega^2}{2} = \frac{\cos x\omega^2 - \cos x\omega}{2\sin\frac{x}{2}\sin\frac{x\omega}{2}\sin\frac{x\omega^2}{2}}$

$$= \frac{2(\cos x\omega - \cos x\omega^2)}{\sin x + \sin x\omega + \sin x\omega^2}$$

Multiplying both sides by $(\omega^2 - \omega)$ and adding to the corresponding sides of the previous result we have

$$-\frac{x}{2}\left(\cot\frac{x}{2} + \omega^2\cot\frac{x\omega}{2} + \omega\cot\frac{x\omega}{2}\right) = -x.\frac{\cos x + \omega^2\cos x\omega + \omega\cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2}$$

Hence, as before,

$$3\left(B_2\frac{x^2}{\underline{|2}}+B_8\frac{x^8}{\underline{|8}}+B_{14}\frac{x^{14}}{\underline{|14}}+\&c.\right)=x.\frac{\dfrac{x^4}{\underline{|4}}-\dfrac{x^{10}}{\underline{|10}}+\dfrac{x^{16}}{\underline{|16}}-\&c.}{\dfrac{x^3}{\underline{|3}}-\dfrac{x^9}{\underline{|9}}+\dfrac{x^{15}}{\underline{|15}}-\&c.}\quad\ldots\quad(10)$$

Similarly

$$-x\left(\cot\frac{x}{2}+\cot\frac{x\omega}{2}+\cot\frac{x\omega^2}{2}\right)=x.\frac{\cos x+\cos x\omega+\cos x\omega^2-3}{\sin x+\sin x\omega+\sin x\omega^2}$$

and therefore

$$6\left(B_4\frac{x^4}{\underline{|4}}+B_{10}\frac{x^{10}}{\underline{|10}}+B_{16}\frac{x^{16}}{\underline{|16}}+\&c.\right)=x.\frac{\dfrac{x^6}{\underline{|6}}-\dfrac{x^{12}}{\underline{|12}}+\dfrac{x^{18}}{\underline{|18}}-\&c.}{\dfrac{x^3}{\underline{|3}}-\dfrac{x^9}{\underline{|9}}+\dfrac{x^{15}}{\underline{|15}}-\&c.}\quad\ldots\quad(11)$$

Multiplying up and equating coefficients in (9), (10) and (11) as usual, we have,

$$c_3B_{n-3}-c_9B_{n-9}+c_{15}B_{n-15}-=0\ \ldots\qquad\ldots\qquad\ldots\quad(12)$$

the last term being $\frac{n}{6}(-1)^{\frac{n-1}{6}}$, or $\frac{n}{3}(-1)^{\frac{n+1}{6}}$, or $\frac{n}{3}(-1)^{\frac{n-6}{6}}$

Again, dividing both sides in (10) by x and differentiating we have,

$$3\left(\frac{B_2}{\underline{|2}}+7B_8\frac{x^6}{\underline{|8}}+13B_{14}\frac{x^{12}}{\underline{|14}}+\&c\right)=\frac{d}{dx}\left(\frac{\dfrac{x^4}{\underline{|4}}-\dfrac{x^{10}}{\underline{|10}}+\dfrac{x^{16}}{\underline{|16}}-\&c}{\dfrac{x^3}{\underline{|3}}-\dfrac{x^9}{\underline{|9}}+\dfrac{x^{15}}{\underline{|15}}-\&c}\right)$$

$$=1-\frac{\dfrac{x^2}{\underline{|2}}-\dfrac{x^8}{\underline{|8}}+\dfrac{x^{14}}{\underline{|14}}-\&c}{\dfrac{x^3}{\underline{|3}}-\dfrac{x^9}{\underline{|9}}+\dfrac{x^{15}}{\underline{|15}}-\&c}\cdot\frac{\dfrac{x^4}{\underline{|4}}-\dfrac{x^{10}}{\underline{|10}}+\dfrac{x^{16}}{\underline{|16}}-\&c}{\dfrac{x^3}{\underline{|3}}-\dfrac{x^9}{\underline{|9}}+\dfrac{x^{15}}{\underline{|15}}-\&c}.$$

Hence by (9) and (10)

$$3\left(B_2\frac{x^2}{\underline{|2}}+7B_8\frac{x^8}{\underline{|8}}+13B_{14}\frac{x^{14}}{\underline{|14}}+\&c.\right)=x^2+9\left(B_0+B_6\frac{x^6}{\underline{|6}}+B_{12}\frac{x^{12}}{\underline{|12}}+\&c.\right)\times$$
$$\left(B_2\frac{x^2}{\underline{|2}}+B_8\frac{x^8}{\underline{|8}}+B_{14}\frac{x^{14}}{\underline{|14}}+\&c.\right).$$

Equating the coefficients of x^n we have, if $n-2$ is a multiple of 6

$$\frac{n+2}{3}B_n=c_6B_{n-6}B_6+c_{12}B_{n-12}B_{12}+c_{18}B_{n-18}B_{18}+\&c.\ldots\ (13)$$

From (12) the B's can be calculated very quickly and (13) may prove useful in checking the calculations. The number of terms is one-third of that in (4); thus B_{24} is found from B_{18}, B_{12} and B_6.

5. We shall see later on how the B's can be obtained from their properties only. But to know these properties, it will be convenient to calculate a few B's by substituting 0, 2, 4, 6, &c., for n in succession in (12). Thus :—

$$B_0=-1\,;\ B_2=\frac{1}{6}\,;\ B_4=\frac{1}{30}\,;\ B_6=\frac{1}{42}\,;\ B_8-\tfrac{1}{3}B_2=-\frac{1}{45}\,;$$

$$B_{10}-\frac{5}{2}B_4=-\frac{1}{132}\,;\ B_{12}-11B_6=-\frac{4}{455}\,;\ B_{14}-\frac{143}{4}B_8+\frac{B_2}{5}=\frac{1}{120}\,;$$

$$B_{16}-\frac{286}{3}B_{10}+4B_4=\frac{1}{306}\,;\ B_{18}-221B_{12}+\frac{204}{5}B_6=\frac{3}{665}\,;$$

$$B_{20}-\frac{3230}{7}B_{14}+\frac{1938}{7}B_8-\frac{B_2}{7}=-\frac{1}{231}\,;$$

$$B_{22}-\frac{3553}{4}B_{16}+\frac{7106}{5}B_{10}-\frac{11}{2}B_4=-\frac{1}{552}\,;\ \text{and so on.}$$

Hence we have finally the following values :—

$$B_2=\frac{1}{6}\,;\ B_4=\frac{1}{30}\,;\ B_6=\frac{1}{42}\,;\ B_8=\frac{1}{30}\,;\ B_{10}=\frac{5}{66}\,;$$

$$B_{12}=\frac{691}{2730}\,;\ B_{14}=\frac{7}{6}\,;\ B_{16}=\frac{3617}{510}\,;\ B_{18}=\frac{43867}{798}\,;$$

$$B_{20}=\frac{174611}{330}\,;\ B_{22}=\frac{854513}{138}\,;\ B_{24}=\frac{236364091}{2730}\,;$$

$$B_{26}=\frac{8553103}{6}\,;\ B_{28}=\frac{23749461029}{870}\,;$$

$$B_{30}=\frac{8615841276005}{14322}\,;\ B_{32}=\frac{7709321041217}{510}\,;$$

$$B_{34}=\frac{2577687858367}{6}\,;$$

$$B_{36}=\frac{26315271553053477373}{1919190}\,;$$

$$B_{38}=\frac{2929993913841559}{6}\,;$$

$$B_{40}=\frac{261116241613172814031}{13530}\,;\ \text{\&c., and}$$

$$B_\infty=\infty.$$

6. It will be observed* that, if n is even excluding 0,

(i) B_n is a fraction and the numerator of B_n/n in its lowest terms is a prime number ... ... (14)

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* See § 12 below.

(ii) the denominator of B_n contains each of the factors 2 and 3 once and only once ... ... (15)

(iii) $2^n\ (2^n-1)\ B_n/n$ is an integer and consequently $2(2^n-1)\ B_n$ is an *odd* integer ... ... (16)

From (16) it can easily be shewn that the denominator of $2(2^n-1)\ B_n/n$ in its lowest terms is the greatest power of 2 which divides n; and consequently, if n is not a multiple of 4, then $4(2^n-1)\ B_n/n$ is an odd integer. ... ... (17)

It follows from (14) that the numerator of B_n in its lowest terms is divisible by the greatest measure of n prime to the denominator, and the quotient is a prime number ... ... (18)

(*a*) Ex: 2 and 3 are the only prime numbers in 12, 24 and 36 and they are found in the denominators of B_{12}, B_{24} and B_{36} and their numerators are prime numbers.

(*b*) 11 is not found in the denominator of B_{22} hence its numerator is divisible by 11; similarly, the numers. of B_{26}, B_{34}, B_{38} are divisible by 13, 17, 19, respectively and the quotients in all cases are prime numbers.

(*c*) 5 is found in the denominator of B_{20}, and not in that of B_{30} and consequently the numerator of B_{30} is divisible by 5 while that of B_{20} is a prime number. Thus we may say that if a prime number appearing in n is not found in the denominator it will appear in the numerator and *vice versa*.

7. Next, let us consider the denominators.

All the denominators are divisible by 6; those of B_4, B_8, B_{12}...by 5 those of B_6, B_{12}, B_{18}...by 7; those of B_{10}, B_{20}, B_{30} ... by 11; But those of B_8, B_{16}, B_{24} are *not* divisible by 9; and those of B_{14}, B_{28} ... are *not* by 15. Hence we may infer that :—

The denominator of B_n is the continued product of prime numbers which are the next numbers (in the natural order) to the factors of n (including unity and the number itself) ... ... (19)

As an example take the denominator of B_{24}. Write all the factors of 24, *viz*: 1,2,3,4,6,8,12, 24. The next numbers to these are 2, 3, 4, 5, 7, 9, 13, 25. Strike out the *composite* numbers and we have the prime numbers 2, 3, 5, 7, 13. And the denominator of B_{24} is the product of 2,3,5,7,13 = 2730.

It is unnecessary to write the *odd* factors of n except unity, as the next numbers to these are even and hence composite.

The following are some further examples :

Even factors of n and unity.		Denominator of B_n.
B_2	... 1,2	$2.3=6$
B_6	... 1,2,6	$2.3.7=42$
B_{12}	... 1,2,4,6,12	$2.3.5.7.13=2730$
B_{20}	... 1,2,4,10,20	$2.3.5.11=330$
B_{30}	... 1,2,6,10,30	$2.3.7.11.31=14322$
B_{42}	... 1,2,6,14,42	$2.3.7.43=1806$
B_{56}	... 1,2,4,8,14,28,56 ...	$2.3.5.29=870$
B_{72}	... 1,2,4,6,8,12,18,24,36,72 ...	$2.3.5.7.13.19.37.73=140100870$
B_{90}	... 1,2,6,10,18,30,90 ...	$2.3.7.11.19.31=272118$
B_{110}	... 1,2,10,22,110	$2.3.11.23=1518$

8. Again taking the fractional part of any B and splitting it into partial fractions, we see that :—

The fractional part of $B_n=(-1)^{\frac{n}{2}}$ { the sum of the reciprocals of the prime factors of the denominator of B_n-1 } (20)

Thus the fractional part of $B_{16}=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}-1=\frac{47}{510}$;

that of $B_{22}=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{23}=\frac{17}{138}$

that of $B_{28}=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{29}-1=\frac{59}{870}$; and so on.

9. It can be inferred from (20) that :—

If G be the G.C.M. and L the L.C.M of the denominators of B_m and B_n, then L/G is the denominator of $[B_m-(-1)^{\frac{m-n}{2}}B_n]$, and hence, if the denominators of B_m and B_n are equal then $[B_m-(-1)^{\frac{m-n}{2}}B_n]$ is an integer* ... ... (21)

Example : $B_{24}-B_{12}$ and $B_{32}-B_{16}$ are integers, while the denominator of $B_{10}+B_{20}$ is 5.

It will be observed that

(1) If n is a multiple of 4, then the numerator of $\left(B_n-\frac{1}{30}\right)$ in its lowest terms is divisible by 20 ; but if n is not a

multiple of 4 then that of $\left(\frac{B_n}{n}-\frac{1}{12}\right)$ in its lowest terms is divisible by 5 ... ... (22)

(2) If n is any integer, then

$$2(2^{4n+2}-1)\frac{B_{4n+2}}{2n+1},\ 2(2^{8n+4}-1)\frac{B_{8n+4}}{2n+1},\ 2(2^{12n+6}-1)\frac{B_{12n+6}}{2n+1}$$

are integers of the form $30p+1$ ... ... (23)

10. If a B is known to lie between certain limits then it is possible to find its exact value from the above properties.

Suppose we know that B_{22} lies between 6084 and 6244; its exact value can be found as follows:

The fractional part of $B_{22}=\frac{17}{138}$ by (20), also B_{22} is divisible by 11 by (18). And by (22) $B_{22}-\frac{11}{6}$ must be divisible by 5. To satisfy these conditions B_{22} must be either $6137\frac{17}{138}$ or $6192\frac{17}{138}$.

But according to (18) the numerator of B_{22} should be a *prime* number after it is divided by 11; and consequently B_{22} must be equal to $6192\frac{17}{138}$ or $\frac{854513}{138}$, since the numerator of $6137\frac{17}{138}$ is not only divisible by 11 but also by 7 and 17.

11. It is known (Edward's Diff. Cal., Ch. V, Ex. 29) that

$$B_n=\frac{n}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c.,\right)$$

or $$\frac{2\lfloor n}{(2\pi)^n}=B_n\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\&c. \quad \ldots \quad \ldots \quad (24)$$

where 2,3,5, ..., are prime numbers ... ...

Also $$\frac{B_n}{2n}=\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}dx \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

For $$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}dx=\int_0^\infty x^{n-1}(e^{-2\pi x}+e^{-4\pi x}+\ldots,)dx$$

$$=\frac{\lfloor n-1}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c.,\right)=\frac{B_n}{2^n} \text{ by (24)}$$

In a similar manner

$$\int_0^\infty \frac{x^n}{(e^{\pi x}-e^{-\pi x})^2}dx=\frac{B_n}{4\pi}$$

and $$\int_0^\infty x^{n-1}\log(1-e^{-2\pi x})dx=-\frac{\pi B_n}{n(n-1)} \quad \ldots \quad (26)$$

Take logarithms of both sides in (24) and write for $\log_e \underline{|n}$ the well-known expansion of $\log_e \Gamma(n+1)$, as in Carr's Synopsis, *viz.*

$$(n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log (2\pi) + \frac{B_2}{1.2n} - \frac{B_4}{3.4n^3} + \frac{B_6}{5.6n^5} - \dots$$

$$-(-1)^p \frac{B_{2p}\theta}{(2p-1)2^p n^{2p-1}} \quad \dots \quad \dots \quad \dots \quad \dots \quad (27)$$

where θ is a proper fraction, and the last term which stands for the series

$$-\frac{B_{2p}}{(2p-1)2p.n^{2p-1}} + \frac{B_{2+2p}}{(2p+1)(2p+2).n^{2p+1}} + \&c.,$$

$$= \frac{1}{\pi} \int_0^\infty \frac{x^{2p-2}}{n^{2p-1}} \log (1-e^{-2\pi x}) dx$$

$$-\frac{1}{\pi} \int_0^\infty \frac{x^{2p}}{n^{2p+1}} \log (1-e^{-2\pi x}) dx + \&c.,$$

$$= \frac{1}{\pi} \int_0^\infty \left(\frac{x^{2p-2}}{n^{2p-1}} - \frac{x^{2p}}{n^{2p+1}} - \&c. \right) \log (1-e^{-2\pi x}) dx.$$

$$= \frac{1}{\pi} \int_0^\infty \frac{x^{2p-2}}{n^{2p-3}(n^2+x^2)} \log (1-e^{-2\pi x})\, dx.$$

$$= \int_0^\infty \frac{x^{2p-2} \log (1-e^{-2\pi nx})}{\pi (1+x^2)}\, dx.$$

We can find the integral part of B_n and since the fractional part can be found, as shown in § 8, the exact value of B_n is known. Unless the calculation is made to depend upon the values $\log_{10} e$, $\log_e 10$, π &c. whose values are known to a great number of decimal places, we should have to find the logarithms of certain numbers whose values are not found in the tables to as many places of decimals as we require. Such difficulties are removed by the method given in § 13.

Results (14) to (17), (20) and (21) can be got from the following result:

$$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \dots\dots\dots$$

$$-\frac{1}{x} - \frac{1}{6(x^3-x)} + \frac{1}{5(x^5-x)} + \frac{1}{7(x^7-x)} + \frac{1}{11(x^{11}-x)} + \dots$$

$$= \frac{1}{x^{13}} - \frac{7}{x^{17}} + \frac{55}{x^{19}} - \frac{529}{x^{21}} + \quad \dots \quad \dots \quad \dots \quad \dots \quad (28)$$

where 5, 7, 11, 13, are prime numbers above 3.

If we can prove that the L.H.S. of (28) can be expanded in ascending powers of $\frac{1}{x}$ with integral coeffts, then (20) and (21) are at once deduced as follows:—

From (27) we have

$$\frac{d^2\log\Gamma(n+1)}{d\,n^2}=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\&c.$$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{B_2}{n^3}-\frac{B_4}{n^5}+\frac{B_6}{n^7}-\frac{B_8}{n^9}+\ldots\ldots\ldots\ldots-(-)^p\frac{B_{2p}\theta}{n^{2p+1}} \quad \ldots \quad (29)$$

which stands for the series

$$\frac{B_{2p}}{n^{2p+1}}-\frac{B_{2p+2}}{n^{2p+3}}+\&c.$$

$$=4\pi\int_0^\infty\frac{x^{2p}}{n^{2p+1}(e^{\pi x}-e^{-\pi x})^2}dx$$

$$-4\pi\int_0^\infty\frac{x^{2p+2}}{n^{2p+3}(e^{\pi x}-e^{\pi x})^2}dx+\&c.$$

$$=\pi\int_0^\infty\left(\frac{x^{2p}}{n^{2p+1}}-\frac{x^{2p+2}}{n^{2p+3}}+\ \&c.\right)\frac{dx}{\sinh^2\pi x}$$

$$=\pi\int_0^\infty\frac{x^{2p}}{n^{2p-1}(n^2+x^2)}\,\frac{dx}{\sinh^2\pi x}=\int_0^\infty\frac{\pi x^{2p}}{(1+x^2)\sinh^2\pi n x}dx$$

Substituting the result of (29) in (28) we see that

$$\frac{B_2}{x^3}-\frac{B_4}{x^5}+\frac{B_6}{x^7}-\ \ldots\ -\frac{1}{6(x^3-x)}+\frac{1}{5(x^5-x)}+\frac{1}{7(x^7-x)}+\frac{1}{11(x^{11}-x)}+\cdots$$

where 5, 7, 11, &c., are prime numbers can be expanded in ascending powers of $1/x$ with integral coeffts.

Therefore $B_2-\frac{1}{6}$, $-B_4-\frac{1}{6}+\frac{1}{5}$, $B_6-\frac{1}{6}+\frac{1}{7}$, $-B_8-\frac{1}{6}+\frac{1}{5}$, $B_{10}-\frac{1}{6}+\frac{1}{11}$, &c which are the coeffts of $1/x^3$, $1/x^5$, $1/x^7$, &c., are integers.

Writing $\frac{1}{2}+\frac{1}{3}-1$ for $-\frac{1}{6}$ we get the results of (20) and (21).

Again changing n to $\frac{n}{2}$ in (29) and subtracting half of the result from (29) we have

$$\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}-\cdots=\frac{1}{2n^2}-\frac{(2^2-1)\,B_2}{n^3}+\frac{(2^4-1)\,B_4}{n^5}$$

$$-\frac{(2^6-1)\,B_6}{n^7}+\ldots\ldots\ldots+(-1)^p(2^{2p}-1)\frac{B_{2p}\theta}{n^{2p+1}} \quad \ldots \quad (30)$$

where θ is a proper fraction, and also by (29);

$$(2^{2p}-1)\frac{B_{2p}\theta}{n^{2p+1}}=\int_0^\infty\frac{\pi x^{2p}\cosh\,\pi n x}{(1+x^2)\,\sinh^2\pi n x}\,dx$$

Thus we see that, if we can prove that twice the L.H.S. of (30) can be expanded in ascending powers of $1/n$ with integral coeffects. Then the second part of (16) is at once proved.

Again from (27) we have

$$\frac{d\log\Gamma(n+1)}{dn}=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots+\tfrac{1}{n}-c$$

$$=\log n+\frac{1}{2n}-\frac{B_2}{2n^2}+\frac{B_4}{4n^4}-\frac{B_6}{6n^6}+\frac{B_8}{8n^8}-\ldots+(-1)^p\frac{B_{2p}\theta}{2p.n^{2p}} \quad \ldots \quad (31)$$

where θ is a proper fraction; and also by (25)

$$\frac{B_{2p}\theta}{2p.n^{2p}}=\int_0^\infty\frac{2x^{2p-1}}{(1+x^2)(e^{2\pi nx}-1)}dx$$

from which it can easily be shewn that

$$\frac{1}{n+2}-\frac{1}{n+4}+\frac{1}{n+6}-\frac{1}{n+8}+\frac{1}{n+10}-\&c$$

$$=\frac{1}{2n}-2(2^2-1)\frac{B_2}{2n^2}+2^3(2^4-1)\frac{B_4}{4n^4}$$

$$-2^5(2^6-1)\frac{B_6}{6n^6}+\ldots+(-1)^p2^{2p-1}(2^{2p}-1)\frac{B_{2p}\theta}{2p.n^{2p}}- \quad \ldots \quad (32)$$

where θ is a proper fraction; and also by (31)

$$2^{2p-1}(2^{2p}-1)\frac{B_{2p}\theta}{2p.n^{2p}}=\int_0^\infty\frac{(2x)^{2p-1}}{2(1+x^2)\sinh\frac{\pi nx}{2}}dx$$

From the above theorem we see that, if we can prove that

$$2\left(\frac{1}{n+2}-\frac{1}{n+4}+\frac{1}{n+6}-\&c.,\right)$$

can be expanded in ascending powers of $1/n$ with integral coeffts. then the first part of (16) is at once got.

13. The first few digits, and the number of digits in the integral part as well as in the numerator of B_n, can be found from the approximate formula :—

$$\log_{10} B_n=(n+\tfrac{1}{2})\log_{10} n-1.2324743503n+.700120$$

the true value being greater by about $\cdot 0362\ n$ when n is great ... (33)

This formula is proved as follows:

Taking logarithms of both sides in (24)

$$\log_e B_n=(n+\tfrac{1}{2})\log_e n-n(1+\log_e 2\pi)+\tfrac{1}{2}\log_e 8\pi+\frac{1}{2^n}\text{ nearly.}$$

Multiplying both sides by $\log_{10} e$ or $\cdot 4342944819$, and reducing we can get the result.

14. Changing n to $n-2$ in (24) and taking the ratio sult the two results, we have

$$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \left(1 - \frac{2^2-1}{2^n-1}\right) \left(1 - \frac{3^2-1}{3^n-1}\right) \left(1 - \frac{5^2-1}{5^n-1}\right) \&c \; \ldots \; (34)$$

where 2, 3, 5, &c. are prime numbers.

Hence we see that

$$\frac{B_n}{B_{n-2}} \text{ approaches } \frac{n(n-1)}{4\pi^2}$$

very rapidly as n becomes greater and greater ... ... (35)

From the value of π, *viz*, 3·14159, 26535, 89793, 23846, 26433, 83279, 50288, 41971, 69399.................the integral part of any B can be found from the previous B; and from the integral part the exact value can at once be written by help of (20) as follows :—

Approximate Ratio of any B to previous B,	lies between.*	Hence the exact value is	
B_2	0 & 1 ...	$1-\frac{1}{2}-\frac{1}{3}$	$=\frac{1}{6}$
$B_4=\frac{3\cdot 4}{4\pi^2}B_2$ ...	0 & 1 ...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1$	$=\frac{1}{30}$
$B_6=\frac{5\cdot 6}{4\pi^2}B_4$ ...	0 & 1 ...	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}$	$=\frac{1}{42}$
$B_8=\frac{7\cdot 8}{4\pi^2}B_6$ ...	0 & 1 ...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1$	$=\frac{1}{30}$
$B_{10}=\frac{9\cdot 10}{4\pi^2}B_8$ ...	0 & 1 ...	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{11}$	$=\frac{5}{66}$
$B_{12}=\frac{11\cdot 12}{4\pi^2}B_{10}$ ...	0 & 1 ...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}-1$	$=\frac{691}{2730}$
$B_{14}=\frac{13\cdot 14}{4\pi^2}B_{12}$ ...	0 & 2 ...	$2-\frac{1}{2}-\frac{1}{3}$	$=\frac{7}{6}$
$B_{16}=\frac{15\cdot 16}{4\pi^2}B_{14}$ ...	7 & 8 ...	$6+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}$	$=\frac{3617}{510}$
$B_{18}=\frac{17\cdot 18}{4\pi^2}B_{16}$ ...	54 & 55 ...	$56-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}-\frac{1}{19}$	$=\frac{43867}{798}$
$B_{20}=\frac{19\cdot 20}{4\pi^2}B_{18}$ ...	529 & 530 ...	$528+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}$	$=\frac{174611}{330}$
&c. &c.			

* These integral limits are got from a rough calculation of any B from the preceding B by the formula given in the first column.

15. **From the preceding theorems we know some of the properties of B_n for all positive even values of n. As an example let us take B_{444} :**

The fractional part of $B_{444}=\dfrac{23975417}{90709710}$ by (20) The numerator of B_{444} is divisible by 37 and the quotient is a prime number by (18). Again $\log_{10} B_{444}=630.2433$ nearly by (33). Therefore the integral part of B_{444} contains 631 digits the first 4 digits being 1751.

$$\log_{10} N=\log_{10} B_{444}+\log_{10} D,$$

$$=630\ 2433+\log_{10} 90709710=638.2010 \text{ nearly.}$$

Therefore N contains 639 digits the first four digits being 1588.

Again the numerator of $\left(B_{444}-\frac{1}{30}\right)$ is divisible by 20 ; that is the integral

$$\text{part of } B_{444}+\frac{23975417}{90709710}-\frac{1}{30}, \text{ or that of } B_{444}+\frac{698392}{3023657}$$

has a numerator divisible by 20. Therefore the integral part of B_{444} ends with 4 and also the figure next to the last is even.

Hence N ends with 57 and also the third figure from the last is even.

16. Instead of starting with cot x in §§ 2 and 3, we may start with tan or x cosec x and get other similar results.

Thus :

$$\text{(i)}\ \tfrac{4}{3} B_n(2^n-1)=c_6 B_{n-6}(2^{n-6}-1)-c_{12}B_{n-12}(2^{n-12}-1)+\dots$$

$$+\frac{n}{3}(-1)^{\frac{n-2}{6}} \text{ or } \frac{n}{6}(-1)^{\frac{n-4}{6}} \text{ or } \frac{n}{3}(-1)^{\frac{n-6}{6}}\dots\dots\dots \qquad (36)$$

$$\text{(ii)}\ c_3\left(1-\frac{1}{2^{n-4}}\right)B_{n-3}-c_9\left(1-\frac{1}{2^{n-10}}\right)B_{n-9}+c_{15}\left(1-\frac{1}{2_{n-16}}\right)B_{n-15}-\&c.,$$

$$=\tfrac{2}{3}\cdot\frac{n}{2^n}\left\{3.^{\frac{n-1}{2}}+(-1)^{\frac{n-3}{6}} \text{ or } (-1)^{\frac{n+1}{6}} \text{ or } (-1)^{\frac{n+5}{6}}\right\}\ \dots \qquad (37)$$

17. The formulæ obtained in §§ 1, 3, 4 may be called the one interval, two interval and three interval formulæ respectiveiy. The p interval formulæ can be got by taking the pth roots of unity or i according as p is odd or even.

For example, let us take the fifth roots of unity $(1, \alpha, \alpha^2, \alpha^3, \alpha^4,)$ and find the 3 interval formulæ.

Let $\phi(x)=\sin x+\sin x\alpha+\sin x\alpha^2+\sin x\alpha^3+\sin x\alpha^4$

$$=5\left(\frac{x^5}{\underline{|5}}-\frac{x^{15}}{\underline{|15}}+\frac{x^{25}}{\underline{|25}}-\&c.\right)$$

Then it can easily be shown that

$$16 \sin x \sin x\alpha \sin x\alpha^2 \sin x.\alpha^3 \sin x\alpha^4.$$

$$= \phi(2x) - \phi(2x.\overline{\alpha+\alpha^4}) - \phi(2x.\overline{\alpha^2+\alpha^3})$$

$$= \phi(2x) + \phi(x\overline{1+\sqrt{5}}) + \phi(x\overline{1-\sqrt{5}})$$

Taking logarithms and dfferentiating both sides we have

$$5\left(B_0 + B_{10}\frac{x^{10}}{\underline{|10}} + B_{20}\frac{x^{20}}{\underline{|20}} + B_{30}\frac{x^{30}}{\underline{|30}} \quad \&c.\right)$$

$$= -x \frac{\dfrac{x^4}{\underline{|4}}(1+\alpha_5) - \dfrac{x^{14}}{\underline{|14}}(1+\alpha_{15}) + \&.}{\dfrac{x^5}{\underline{|5}}(1+\alpha_5) - \dfrac{x^{15}}{\underline{|15}}(1+\alpha_{15}) + \&c.} \quad \ldots \quad \ldots (38)$$

where $\alpha_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, so that $\alpha_n\alpha_m = \alpha_{n+m} + (-1)^n\alpha_{m-n}$

Similarly,

(i) $$5\left(B_2\frac{x^2}{\underline{|2}} + B_{12}\frac{x^{12}}{\underline{|12}} + B_{22}\frac{x^{28}}{\underline{|22}} + \&c.\right)$$

$$= x.\frac{\dfrac{x^6}{\underline{|6}}(1+\alpha_7) - \dfrac{x^{16}}{16}(1+\alpha_{17}) + \&c,}{\dfrac{x^5}{\underline{|5}}(1+\alpha_5) - \dfrac{x^{15}}{\underline{|15}}(1+\alpha_{15}) + \&c.} \quad \ldots \quad \ldots (39)$$

(ii) $$5\left(B_4\frac{x^4}{\underline{|4}} + B_{14}\frac{x^{14}}{\underline{|14}} + B_{24}\frac{x^{24}}{\underline{|24}} + \&c.\right)$$

$$= x \frac{\dfrac{x^8}{\underline{|8}}(a_7-1) - \dfrac{|x^{18}}{\underline{|18}}(a_{17}-1) + \&c.}{\dfrac{x^5}{\underline{|5}}(1+a_5) - \dfrac{x^{15}}{\underline{|15}}(1+a_{15}) + \&c.,} \quad \ldots \quad \ldots (40)$$

(iii) $$10\left(B_6\frac{x^6}{\underline{|6}} + B_{16}\frac{x^{16}}{\underline{|16}} + B_{26}\frac{x^{26}}{\underline{|26}} + \&c,\right)$$

$$= x.\frac{\dfrac{x^{10}}{\underline{|10}}(a_{10}-3) - \dfrac{x^{20}}{\underline{|20}}(a_{20}-3) + \dfrac{x^{30}}{\underline{|30}}(a_{30}-3) - \&c.,}{\dfrac{x^5}{\underline{|5}}(1+a_5) - \dfrac{x^{15}}{\underline{|15}}(1+a_{15}) + \dfrac{x^{25}}{\underline{|25}}(1+a_{25}) - \&c.,} \quad \ldots \quad \ldots (41)$$

(iv) $$5\left(B_8\frac{x^8}{\underline{|8}} + B_{18}\frac{x^{18}}{\underline{|18}} + B_{28}\frac{x^{28}}{\underline{|28}} + \&c.,\right)$$

$$= x.\frac{\dfrac{x^{12}}{\underline{|12}}(a_{11}-1) - \dfrac{x^{22}}{\underline{|22}}(a_{21}-1) + \dfrac{x^{32}}{\underline{|32}}(a_{31}-1) - \&c.,}{\dfrac{x^5}{\underline{|5}}(1+a_5) - \dfrac{x^{15}}{\underline{|15}}(1+a_{15}) + \dfrac{x^{25}}{\underline{|25}}(1+a_{25}) - \&c.,} \quad (42)$$

Again from

$$16\cos x\cos x\alpha\cos x\alpha^2\cos x\alpha^3\cos x\alpha^4=1+\psi[2x]+\psi[2x(\alpha+\alpha^4)]+\psi[2x(\alpha^2+\alpha^3)],$$

where $\psi(x)=\cos x+\cos x\alpha+\cos x\alpha^2+\cos x\alpha^3+\cos x\alpha^4$

$$=5\left(1-\frac{x^{10}}{\lfloor 10}+\frac{x^{20}}{\lfloor 20}-\frac{x^{30}}{\lfloor 30}+\cdots\right),$$

and similar relations we can get many other identities.

18. The four interval formulae can be got from the following identities:

If $a_n=\left(1+\frac{1}{\sqrt{2}}\right)^n+\left(1-\frac{1}{\sqrt{2}}\right)^n$ and $b_n=\left(1+\frac{1}{\sqrt{2}}\right)^n-\left(1-\frac{1}{\sqrt{2}}\right)^n$,

so that $a_{m+n}=a_m a_n-a_{m-n}/2^n$; and $b_{m+n}=a_m b_n+b_{m-n}/2^n$; then

(i) $4\left\{B_0+B_8\frac{x^8}{\lfloor 8}+B_{16}\frac{x^{16}}{\lfloor 16}+B_{24}\frac{x^{24}}{\lfloor 24}+\&c.\right\}$

$$=-x.\frac{\frac{x^3}{\lfloor 3}a_2-\frac{x^{11}}{\lfloor 11}a_6+\frac{x^{19}}{\lfloor 19}a_{10}-\&c.}{\frac{x^4}{\lfloor 4}a_2-\frac{x^{12}}{\lfloor 12}a_6+\frac{x^{20}}{\lfloor 20}a_{10}-\&c.} \qquad \cdots \qquad \cdots \qquad (43)$$

(ii) $4\left\{B_2\frac{x^2}{\lfloor 2}+B_{10}\frac{x^{10}}{\lfloor 10}+B_{18}\frac{x^{18}}{\lfloor 18}+\&c.\right\}$

$$=-x.\frac{\frac{x^5}{\lfloor 5}a_3-\frac{x^{13}}{\lfloor 13}a_7+\frac{x^{21}}{\lfloor 21}a_{11}-\&c.}{\frac{x^4}{\lfloor 4}a_2-\frac{x^{12}}{\lfloor 12}a_6+\frac{x^{20}}{\lfloor 20}a_{10}-\&c.} \qquad \cdots \qquad \cdots \qquad (44)$$

(iii) $4\sqrt{2}\left\{B_4\frac{x^4}{\lfloor 4}+B_{12}\frac{x^{12}}{\lfloor 12}+B_{20}\frac{x^{20}}{\lfloor 20}+\&c.\right\}$

$$=x.\frac{\frac{x^7}{\lfloor 7}b_3-\frac{x^{15}}{\lfloor 15}b_7+\frac{x^{23}}{\lfloor 23}b_{11}-\&c.}{\frac{x^4}{\lfloor 4}a_2-\frac{x^{12}}{\lfloor 12}a_6+\frac{x^{20}}{\lfloor 20}a_{10}-\&c.} \qquad \cdots \qquad \cdots \qquad (45)$$

(iv) $4\sqrt{2}\left\{B_6\frac{x^6}{|\underline{6}}+B_{14}\frac{x^{14}}{|\underline{14}}+B_{22}\frac{x^{22}}{|\underline{22}}+\&c.\right\}$

$$=x.\frac{\frac{x^9}{|\underline{9}}b_4-\frac{x^{17}}{|\underline{17}}b_8+\frac{x^{25}}{|\underline{25}}b_{12}-\&c.}{\frac{x^4}{|\underline{4}}a_2-\frac{x^{12}}{|\underline{12}}a_6+\frac{x^{20}}{|\underline{20}}a_{10}-\&c.} \qquad \ldots \qquad \ldots \qquad (46)$$

SHORT NOTES.

On Steiner's Tricusp:

The *orthopole* of any chord TT′ of the circumcircle of a triangle ABC is the intersection of the Simson lines of T and T′ (*Lemoine's Theorem*: J. I. M. S., Vol. I, p. 228).

Now, let T′ conicide with T. Then TT′ becomes the tangent at T to the circumcircle. Also, since the Simson lines of T, T′ are tangents to Steiner's tricusp, their intersection coincides in the limit, with the point of contact of the Simson line of T and the tricusp. Hence we have the theorem: 'The locus of the orthopoles of tangents to the circumcircle of a triangle ABC is the three-cusped hypocycloid which is the envelope of the pedal lines of the triangle.'

V. Ramaswami Aiyar.

Pedals and Envelopes.

In Vol. II, No. 3, of this Journal Mr. M. T. Naraniengar has discussed the properties of some *associated curves* called by him *Root-Pedals.* Long before the publication of Mr. Naraniengar's article, I had drawn up a note on 'Pedals and Envelopes', inspired by Mr. V. Ramaswami Iyer, in which I had considered all the properties dealt with in §§ 1—9 of the said article and in addition the following:—

(1) *The* α-*pedal of a curve*:

Draw a straight line from the origin making always a constant angle α with the variable tangent to a curve. The locus of the intersection of the line and the tangent may be called the *oblique* pedal or the α-pedal of the curve, as distinguished from the *right* or first positive pedal.

(2) The α-pedal and the right pedal are similar curves; the former is, in fact, the right pedal turned through $(\pi/2-\alpha)$ and each radius vector lengthened in the ratio cosec α.

(3) The α-pedal of the α-pedal may be called the 2nd α—pedal; and so on, we may have the 1st, 2nd,............*kth* α-pedal of a curve. If r, θ be the coordinates of any point on a curve, the corodinates (r_n, θ_n) of the corresponding point on the nth α-pedal will be found to be given by the equations

$$\frac{r_n}{r}=\left(\frac{\sin\phi}{\sin\alpha}\right)^n \text{ and } \theta_n=\theta+n(\phi-\alpha)$$

where $\cot\phi=\frac{1}{r}\frac{dr}{d\theta}$.

(4) If the nth α-pedal of a curve be defined by these equations we can, as in the case of the right pedal, extend the definition to the cases where n is factional or negative: and the geometric interpretation is suggested by the equations as in the case of Mr. Naraniengar's root—pedals.

Also, the nth α-pedal is similar to the nth right-pedal, the ratio of similitude being cosec $^n\alpha$.

(5) Further, it can be proved, as for right-pedals, that the mth α-pedal of the nth α-pedal is the $(m+n)^{th}$ α-pedal whatever m and n may be.

(6) The β-pedal of the α-pedal of a curve can be proved to be the same as the α-pedal of the β-pedal, each being identical with the second right-pedal turned through $(\pi-\alpha-\beta)$ and magnified in the ratio cosecα. cosecβ: and so on.

(7) The α-pedal is the envelope of segments of circles described on the radii vectores containing the angle $(\pi-\alpha)$.

(8) Generally, if on each radius vector of a curve as axis, a curve similar to $r^m=a^m \sin(\alpha+m\theta)/\sin\alpha$ be described, the envelope of such curves is the $\left(\frac{1}{m}\right)^{th}$ α-pedal of the given curve. For, let r_1,θ_1 be the coordinates of a point P on the curve $f(r,\theta)=0$ then the equation to the curve described on the radius vector to the point is $r^m=r_1^m \sin(\alpha+m\overline{\theta-\theta_1})$/We have to find the envelope of this, where the parameters r_1, θ_1 are connected by the equation $f(r_1\ \theta_1)=0$.

Differentiating with respect to θ_1 we have

$$\frac{m}{r_1}\frac{dr_1}{d\theta_1}-\frac{\cos(\alpha+m\overline{\theta-\theta_1})}{\sin(\alpha+m\overline{\theta-\theta_1})}.\ m=0$$

$$\cot(\alpha+m.\overline{\theta-\theta_1})=\frac{1}{r_1}\frac{dr_1}{d\theta_1}=\cot\phi_1,$$

where ϕ_1 is the 'ϕ' at the point r_1,θ_1 of the curve.

Thus $$\alpha+m(\theta-\theta_1)=\phi_1$$

$\therefore$ $$\theta=\theta_1+(\phi-\alpha)/m \qquad (1)$$

and $$r^m/r_1^m=\sin\phi_1/\sin\alpha$$

$\therefore$ $$r/r_1=(\sin\phi_1/\sin\alpha)^{\frac{1}{m}} \qquad (2)$$

Equations (1) & (2) define the $(1/m)^{th}$ α-pedal by §. 3

Cor. 1. The nth α-pedal of the point (r_1,θ_1) regarded as a curve is

$$r^{1/n}=r_1^{1/n}\sin\left(\alpha+\frac{\theta-\theta_1}{n}\right)/\sin\alpha.$$

For, this represents the curve similar to

$$r^{1/n}=a^{1/n}\sin(\alpha+\theta/n)/\sin\alpha,$$

described on the radius vector to the point.

Cor. 2. The nth right-pedal of the point (a, o) is

$$r^{1/n} = a^{1/n} \cos \theta/n.$$

(9) The preceding theorem may be stated thus:

If a point P describe a curve C, the nth α-pedal of the point P envelopes the nth α-pedal of C.

P. V. SESHU AIYAR.

A Trigonometrical Note

1. Given that, in a plane triangle ABC,

$$a = b \cos C + c \cos B,\ b = c \cos A + a \cos C,\ c = a \cos B + b \cos A$$

we may deduce that

$$a/\sin A = b/\sin B = c/\sin C,$$

as follows.

By eliminating a, b, c from the given equations, we get

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1 = 0 \qquad (1)$$

Also from the first two equations, we find

$$\frac{a}{\cos A \cos C + \cos B} = \frac{b}{\cos B \cos C + \cos A} = \frac{c}{1 - \cos^2 C}$$

Square and reduce the first two denominators by using (1); then

$$\frac{a^2}{(1-\cos^2 A)(1-\cos^2 C)} = \frac{b^2}{(1-\cos^2 B)(1-\cos^2 C)} = \frac{c^2}{(1-\cos^2 C)^2}$$

Hence, $$\frac{a^2}{\sin^2 A} = \frac{b^2}{\sin^2 B} = \frac{c^2}{\sin^2 C}$$

i.e. $$a/\sin A = b/\sin B = c/\sin C.$$

2. The process in § 1 may be presented in a slightly different, but more interesting form, thus.

Let the co-factors of the several elements of the determinant

$$\begin{vmatrix} -1, & \cos C, & \cos B \\ \cos C, & -1, & \cos A \\ \cos B, & \cos A, & -1 \end{vmatrix}$$

be denoted by A_{11} A_{12}......, the first figure in each suffix standing for the row and the second for the column relating to any element. Then by taking every two of the three given relations we obtain

$$a/A_{11} = b/A_{12} = c/A_{13},$$
$$a/A_{21} = b/A_{22} = c/A_{23},$$
$$a/A_{31} = b/A_{32} = c/A_{33}.$$

But, obviously, $A_{12} = A_{21}$, &c. Hence, by multiplication two at a time

$$a^2/A_{11} = b^2/A_{22} = c^2/A_{33},$$

that is, $$a^2/(1-\cos^2 A) = b^2/(1-\cos^2 B) = c^2/(1-\cos^2 C)$$

or, $$a/\sin A = b/\sin B = c/\sin C.$$

3. Next, consider a *tetrahedron* ABCD.

If a,b,c,d denote the areas of the plane faces. We have, by projection

$$a = b\cos(ab) + c\cos(ac) + d\cos(ad)$$

and three more similar relations.

Take the determinant

$$\begin{vmatrix} -1, & \cos(ab), & \cos(ac), & \cos(ad) \\ \cos(ba), & -1, & \cos(bc), & \cos(bd) \\ \cos(ca), & \cos(cb), & -1, & \cos(cd) \\ \cos(da), & \cos(db), & \cos(dc), & -1 \end{vmatrix}$$

Using A_{11}, A_{12}...as before and taking every three of the four equations of projection, we find

$$a/A_{11} = b/A_{12} = c/A_{13} = d/A_{14}$$
$$a/A_{21} = b/A_{22} = c/A_{23} = d/A_{24}$$
$$a/A_{31} = b/A_{32} = c/A_{33} = d/A_{34}$$
$$a/A_{41} = b/A_{42} = c/A_{43} = d/A_{44}$$

Remembering that, $A_{12} = A_{21}$, &c., we easily deduce

$$a^2/A_{11} = b^2/A_{22} = c^2/A_{33} = d^2/A_{44}.$$

Now, the *second staudtian* of the solid angle A is obviously proportional to $\sqrt{A_{11}}$.

Hence, a, b, c, d are proportional to the *second staudtians* of the solid angles at the opposite vertices.

4. Extension of the theorem to the nth dimension, and its proof follow at once.

M. T. NARANIENGAR.

The Face of the Sky for December 1911 and January 1912.

Sidereal time at 8 p.m.

Date.	December. H.	M.	S.	January. H.	M.	S.
1	0	37	18	2	29	31
8	1	4	54	3	7	7
15	1	32	30	3	34	43
22	2	0	6	4	2	19
29	2	27	42	4	29	55

From this table the constellations visible during the evenings can be ascertained by a reference to their position as given in star-charts.

Phases of the Moon.

	December.				January.			
	Date.		H.	M.	Date.		H.	M.
Full Moon ...	6	...	8	22 A.M.	4	...	8	0 A.M.
Last Quarter ...	12	...	11	16 P.M.	11	...	2	13 „
New Moon ...	20	...	9	10 „	19	...	5	40 „
First Quarter ...	29	...	0	18 A.M.	27	...	3	21 „

Sun.

The Sun enters capricornus on December 23rd, at 4-24 A.M.

Planets.

Mercury is an evening star in December and a morning star in January. It is in retrograde motion from December 16 up to January 5. It attains its greatest elmgation (20°58′E) on Dec. 8 at 0-30 A.M.

Venus is a morning star in these months. It is in conjunction with the moon on Dec. 16 at 8-28 P.M. and on Jan. 9 at 10-56 P.M. and with Jupiter on Jan. 10 at 2-30 A.M.

Mars continues to retrograde. It is still in good position though not so favorable as in November. Its diameter diminishes from 18·32″ in the middle of November to 13·48″ at the end of the year. It is in conjunction with the moon on Dec. 5 at 9-25 A.M and on 29 January at 7-28 A.M.

Ceres is in Cancer at the end of the year about 7° E. of Pollux (β Geminorum) and 5° N.N W. of γ Cancri and should be seen by the aid of a telescope late in the evening. She is of the 7th magnitude in December.

Vesta is in conjunction with the moon on Dec. 11 at 4-30 P.M.

Jupiter is a morning star in these months. It is in conjunction with the moon on Dec. 18 at 6-32 P.M.

Saturn continues to retrograde till Jan. 16 when it resumes direct motion. It is in conjunction with the moon on Dec. 4 at 1-9 P.M. and on Jan. 27 at 5-40 P.M.

Neptune is in conjunction with the moon on Dec. 8 at 10-52 P.M.

V. Ramesam.

SOLUTIONS.

Question 142.

(J. C. SWAMINARAYAN, M. A.) :—Solve completely the differential equation $x\frac{d^2y}{dx^2}+(2y-x)\frac{dy}{dx}=y$, and express its complete primitive in the form of a series in ascending powers of x.

Solution by H. Br.

Integrating $$x\frac{d^2y}{dx^2}+(2y-x)\frac{dy}{dx}=y,$$

$$x\frac{dy}{dx}+y^2-y-xy=\text{constant}=k(1-k)\text{ say,}$$

Let $$y=\frac{x}{v}\cdot\frac{dv}{dx}.$$

Substituting $$x^2\left(\frac{d^2v}{dx^2}-\frac{dv}{dx}\right)=k(1-k)v.$$

If $v=a(v_1+cv_2)$, be the complete primitive of this equation,

$$y=\frac{\frac{dv_1}{dx}+c\frac{dv_2}{dx}}{v_1+c\,v_2},$$

the constants of integration being c and k.

The forms of v_1, v_2 vary according to the value of k. In general, if $2k$ is not unity or an even integer, the ordinary methods give

$$v_1=f(k),\text{ and }v_2=f(1-k),$$

where $$f(\mu)=x^\mu\sum_{n=0}^{\infty}\frac{\Gamma(\mu+n)}{\Gamma(2\mu+n)}\cdot\frac{x^n}{n!}$$

Question 247.

(R. SRINIVASAN, M.A.) :—At the point of contact (x,y) of a curve with its circle of curvature, $\frac{d^4x}{ds^4}$ differs from the corresponding quantity for the circle by

$$2\frac{d^2x}{ds^2}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{dx}{ds}\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\frac{dx}{ds}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}$$

Solution and correction by N. Sunkara Aiyar, B.A.

The difference ought to be

$$2\frac{d^2y}{ds^2}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{dy}{ds}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\frac{dx}{ds}\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

or more simply

$$\frac{3}{\rho}\frac{dx}{ds}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{dy}{ds}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

For $$\frac{dx}{ds}=\cos\psi.$$

$\therefore$ $$\frac{d^2x}{ds^2}=-\sin\psi\frac{d\psi}{ds}=-\frac{\sin\psi}{\rho}.$$

and finally $$\frac{d^4x}{ds^4}=\frac{\sin\psi}{\rho^3}-3\frac{\cos\psi}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right)-\sin\psi\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$$
$$=\frac{\sin\psi}{\rho^3}-3\frac{1}{\rho}\frac{dx}{ds}\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{dy}{ds}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

We know that $\frac{dx}{ds}$, $\frac{dy}{ds}$ and ρ are the same for the circle of curvature and the curve, and that ρ being constant its derivatives with respect to s vanish.

Hence $$\frac{d^4x}{ds^4}=\frac{\sin\psi}{\rho^3}, \text{ for the circle.}$$

Hence the difference between the two quantities is
$$\frac{3}{\rho}\frac{dx}{ds}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{dy}{ds}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

Question 276.

(P. V. Seshu Aiyar B.A., L.T.) :—Solve in positive integers the equations $x+y=a(z+w)$; $bxy=zw$; where a and b are positive integers, and x, y, z, w are the unknowns.

Solution (1) by N. Durai Rajan, (2) by H. Br.

(1) Substitute
$$X=x+y\ ;\quad Z=z+w.$$
$$Y=x-y\ ;\quad W=z-w.$$
$\therefore$ $$X=aZ$$
$$b(X^2-Y^2)=Z^2-W^2.$$
$\therefore$ $$b(a^2Z^2-Y^2)=Z^2-W^2,$$

$\therefore$ Therefore we can write
$$p.\ b(aZ-Y)=q.\ (Z+W)$$
$$q.\ (aZ+Y)=p.\ (Z-W)$$
p and q being positive integers (Hall and Knight Algebra Art 379.)

By cross-multiplication we get:

$W : Z : W=[2pq-abp^2-aq^2] : [q^2-p^2b] : [2ab\,pq-p^2b-q^2)$

Hence
$$x=\lambda.\ p\ (q-ab.p)$$
$$y=\lambda.\ q\ (aq-p)$$
$$z=\lambda.\ pb\ (aq-p)$$
$$w=\lambda.\ q\ (q-ab.\ p)$$

(2) Eliminating y,

$$bx\,[a(z+w)-x]=zw,$$

or $$zw-abx\,(z+w)=-bx^2,$$

or $$(z-abx)(w-abx)=b(a^2b-1)x^2.$$

Let α, β, be any two positive or negative integers, such that $\alpha\beta=b(a^2b-1)$, and let $x=\lambda\mu\nu$, when λ, μ, ν are arbitrarily chosen integers.

Then we have

$$z-abx=\alpha\lambda^2\nu,$$

$$w-abx=\beta\mu^2\nu.$$

Therefore $$z=ab\lambda\mu\nu+\alpha\lambda^2\nu,$$

$$w=ab\lambda\mu\nu+\beta\mu^2\nu,$$

and $$y=(2^2ab-1)\lambda\mu\nu+a\alpha\lambda^2\nu+a\beta\mu^2\nu,$$

while $$x=\lambda\mu\nu.$$

Question 291.

(M. Bhimasena Rao) :—ABC, DEF are two triangles inscribed in a circle. Shew that the triangles formed by the pedal lines of A, B, C with respect to DEF, and by the pedal lines of D, E, F with respect to ABC are inscribed in another circle whose centre is equidistant from, and collinear with, the orthocentres of ABC and DEF.

[This is an extension of Q. 178 by Mr. S. Narayanan, B.A. L.T.]

Solution by the Proposer and S. Purushotham B.A.

Let A′B′C′ and D′E′F′ be the triangles formed by the pedal lines of A,B,C with respect to DEF and by the pedal lines of D, E, F with respect to ABC ; H, K, the orthocentres of ABC and DEF ; and X,Y,Z the points of intersection of KA, KB, KC with B′C′, C′A′, A′B′ respectively.

The angle subtended by any two points on the circle is equal to the angle between their pedal lines. The triangle A′B′C′ is directly similar to ABC. Since KX = XA, etc., XYZ and ABC are homothetic. Therefore XYZ is directly similar to A′B′C′, and being inscribed in A′B′C′ its ortho-centre which is the mid-point of KH is the circumcentre of A′B′C′.

It may be easily shown that the triangles ABC, A′B′C′, as well as DEF, D′E′F′ have the same angle and ratio of similitude. Hence the result required.

If θ is the angle of similitude, the ratio of similitude is ($-\cos\theta$.)

Question 293.

(*J. C. Swaminarayan, M.A.*) :—Shew that any triangle ABC

$$c^n = a^n \cos nB + na^{n-1} b \cos \{ (n-1)B - A \}$$
$$+ \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 \cos \{ (n-2)B - 2A \} \ldots + b^n \cos nA$$

n being a positive integer

Solutions (1) by G. Ramachandran, N. Sankara Aiyar, B.A. and many others.

We have

$$c = a \cos B + b \cos A$$
$$0 = a \sin B - b \sin A$$
$$\therefore \quad c = a(\cos B + i \sin B) + b(\cos A - i \sin A)$$
$$= ae^{iB} + be^{-iA}$$
$$\therefore \quad c^n = a^n e^{inB} + na^{n-1} be^{i\{(n-1)B - A\}}$$
$$+ \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 e^{i\{(n-2)B - 2A\}} + \cdots + b^n e^{inA}$$

Equating real parts, we get

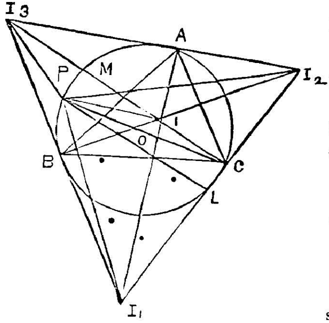
$$c^n = a^n \cos nB + na^{n-1} b \cos \{ (n-1)B - A \}$$
$$+ \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 \cos \{ (n-2)B - 2A \} + \ldots + b^n \cos nA$$

Question 299.

(M. T. NARANIENGAR M.A.) :—With the usual notation shew that in any triangle ABC, II_1, and I_2I_3 subtend supplementary angles at either of the points of the circum circle diametrically opposite to B or C.

Solution by G. Ramachandran and S. Purushottam.

Let P be the point diametrically opposite to C. △ABC is the pedal triangle of $I_1 I_2 I_3$ and the circumcircle of △ ABC is the nine point circle of $I_1 I_2 I_3$.



Let the circle ABC cut I_1I_2 in L and II_3 in M.

Then L and M are the mid points of $I_1 I_2$, II_3.

And PM and PL are ⊥r to II_3 and $I_1 I_2$, since CP is the diameter of ABC.

$\therefore \; I\hat{P}M = I_3\hat{P}M$; and $I_1\hat{P}L = I_2\hat{P}E$.

$\therefore \; I_3\hat{P}I_2 + I\hat{P}I_1 = 2(I_3\hat{P}M + I_2\hat{P}L) = 2\,L\hat{P}M = 2$ right angles, since $L\hat{P}M$ is a right angle.

Question 301 (*a*).

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—Prove that the following expressions are true : the inverse tangents, in all cases, denoting angles between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$:—

$$(1)\sum_1^{\circ} \tan^{-1}\frac{4\sqrt{2}(n^2+3)}{(n^2-1)^2}=2\pi-\tan^{-1}\frac{2\sqrt{2}}{3}$$

$$(2)\sum_1^{\infty} \tan^{-1}\frac{8\sqrt{2}(n^2-6)}{(n^2+2)^2}=\tan^{-1}12\sqrt{2}+\tan^{-1}\frac{6\sqrt{2}}{3}$$

$$(3)\sum_1^{\infty} \tan^{-1}\frac{4n^2}{n^2-4}=\frac{\pi}{2}$$

$$(4)\sum_1^{\infty} \tan^{-1}\frac{16(n^2-8)}{n^2(n^2+16)}=\frac{\pi}{4}$$

$$(5)\sum_1^{\infty} \tan^{-1}\frac{4(n^2-A^2)}{(n^2+A^2)^2-4}=\tan^{-1}\frac{2}{A^2}$$

Solution by H. Br.

In general let it be required to evaluate

$$S=\sum_1^{\infty} \tan_-{}^{1}\frac{an^2+b}{n^4+cn^2+d}$$

We have $iS=$ imaginary part of ϕ, where

$$\phi=\sum_1^{\infty}\log\left(1+\frac{c}{n^2}+\frac{d}{n^4}+\frac{ia}{n^2}+\frac{ib}{n^4}\right)$$

$$=\sum_1^{\infty}\log\left(1-\frac{\lambda^2}{n^2}\right)\left(1-\frac{\mu^2}{n^2}\right)\text{ say, } \lambda, \mu \text{ being complex}$$

$$=\log\frac{\sin\pi\lambda}{\pi\lambda}+\log\frac{\sin\pi\mu}{\pi\mu}$$

$$=\text{real quantities}+\log\ldots-\log\lambda\mu$$

Also $\lambda^2\mu^2=d+ib.$

$\therefore$ $\log\lambda\mu=\frac{1}{2}\log(d+ib)$

If $\sin\pi\lambda.\sin\pi\mu=\alpha+i\beta,$

$$S=\tan^{-1}\frac{\beta}{\alpha}-\frac{1}{2}\tan^{-1}\frac{b}{d}.$$

If a, b, c, d are such that the real parts of λ and μ are integers, $\sin\pi\lambda$ and $\sin\pi\mu$ are both pure imaginaries, and $\tan^{-1}\frac{\beta}{\alpha}$ is a multiple of π.

I may add that I do not obtain the required answers in the first two of the five examples.

Partial solution by R. Srinivasan, M.A.

Q. 301 (5) $\tan^{-1}\dfrac{4(n^2-A^2)}{(n^2+A^2)^2-4}$

$$=\tan^{-1}\frac{2}{(n+iA)^2}+\tan^{-1}\frac{2}{(n-iA)^2}$$

$$=\tan^{-1}\frac{1}{n+iA-1}-\tan^{-1}\frac{1}{n+iA+1}$$

$$+\tan^{-1}\frac{1}{n-iA-1}-\tan^{-1}\frac{1}{n-iA+1}$$

$$\therefore \quad \sum_1^\infty \tan^{-1}\frac{4(n^2-A^2)}{(n^2+A^2)^2-4}=\tan^{-1}\frac{1}{iA}+\tan^{-1}\frac{1}{1+iA}+\tan^{-1}\frac{1}{-iA}$$

$$+\tan^{-1}\frac{1}{1-iA},\ \text{the rest vanishing.}$$

$$=\tan^{-1}\frac{(1+iA)+(1-iA)}{-1+(1+iA)(1-iA)}$$

$$=\tan^{-1}\frac{2}{A^2}$$

301 (4) $\tan^{-1}\dfrac{16(n^2-8)}{(n^2+8)^2-64}=\tan^{-1}\dfrac{8}{(n+i\sqrt{8})}+\tan^{-1}\dfrac{8}{(n-i\sqrt{8})^2}$

$$=\tan^{-1}\frac{1}{\frac{n+i\sqrt{8}}{2}-1}-\tan^{-1}\frac{1}{\frac{n+i\sqrt{8}}{2}+1}$$

$$+\tan^{-1}\frac{1}{\frac{n-i\sqrt{8}}{2}-1}-\tan^{-1}\frac{1}{\frac{u-i\sqrt{8}}{2}+1}$$

as in (5).

$$\therefore \text{L.H.S.}=\tan^{-1}\frac{2}{-1+i\sqrt{8}}+\tan^{-1}\frac{2}{i\sqrt{8}}+\tan^{-1}\frac{2}{n+i\sqrt{8}}+\tan^{-1}\frac{2}{2+i\sqrt{8}}+$$

$$\tan^{-1}\frac{2}{-1-i\sqrt{8}}+\tan^{-1}\frac{2}{-i\sqrt{8}}+\tan^{-1}\frac{2}{1-i\sqrt{8}}+\tan^{-1}\frac{2}{2-i\sqrt{8}}$$

$$=\tan^{-1}\frac{1}{1+\sqrt{2}}+\tan^{-1}\frac{1}{1-i\sqrt{2}}$$

$$=\tan^{-1}\frac{(1+i\sqrt{2})+(1-i\sqrt{2})}{(1+i\sqrt{2})(1-i\sqrt{2})-1}=\tan^{-1}1=\frac{\pi}{4}.$$

Question 304.

(J. C. SWAMINARAYAN, M.A.) :—A particle is repelled from a centre of repulsion O, with a force whose intensity at a distance x is $\frac{\mu}{x^2}-\frac{\lambda}{x^3}$; the particle starts from rest at P, (OP$=a$ and $a\mu>\lambda$). Shew that it will arrive at a point Q in OP produced, (OQ$=b$) in time

$$\frac{a}{2\mu a-\lambda}\left[\sqrt{(2\mu a-\lambda)b^2-2\mu a^2b+a^2\lambda}\right]$$

$$+\frac{\mu a^3}{(2\mu a-\lambda)^{\frac{3}{2}}}\cosh^{-1}\frac{(2\mu a-\lambda)b-\mu a^2}{a(\mu a-\lambda)}$$

Solution by N. Sankara Aiyar, B.A., R. Srinivasan, M.A., S. Narayan B.A., L. T, and othres.

The law of force is given by

$$\frac{d^2x}{dt^2}=\frac{\mu}{x^2}-\frac{\lambda}{x^3}.$$

$$\therefore\left(\frac{dx}{dt}\right)^2=\frac{\lambda}{x^2}-\frac{2\mu}{x}+c=\frac{\lambda-2\mu x+cx^2}{x^2}.$$

When $x=a, \frac{dx}{dt}=0$ & $c=\frac{2\mu}{a}-\frac{\lambda}{a^2}$

$$\therefore\quad\frac{x\,dx.}{\sqrt{cx^2-2\mu x+\lambda}}=dt.$$

Substituting $cx-\mu=z$, we get

$$\frac{(z+\mu)\,dz}{c^{\frac{3}{2}}\sqrt{z^2+c\lambda-\mu^2}}=dt.$$

i.e. $$dt=\frac{z\,dz}{c^{\frac{3}{2}}\sqrt{z^2+c\lambda-\mu^2}}+\frac{\mu\,dz}{c^{\frac{3}{2}}\sqrt{z^2+c\lambda-\mu^2}}$$

Integrating

$$+c_1=\frac{\sqrt{z^2+c\lambda-\mu^2}}{c^{\frac{3}{2}}}+\frac{\mu}{c^{\frac{3}{2}}}\int\sec\theta\,d\theta,$$

where $z^2=(\mu^2-c\lambda)\sec^2\theta$.

$$\text{Therefore } t+c_1=\frac{\sqrt{z^2+c\lambda-\mu^2}}{c^{\frac{2}{3}}}+\frac{\mu}{c^{\frac{3}{2}}}\cosh^{-1}\sec\theta$$

$$=\frac{\sqrt{z^2+c\lambda-\mu^2}}{c^{\frac{3}{2}}}+\frac{\mu}{c^{\frac{3}{2}}}\cosh^{-1}\frac{cx-\mu}{\sqrt{\mu^2-c\lambda}}.$$

Substituting the value of c we get

$$t+c_1=a\left\{\frac{(2\mu a-\lambda)x^2-2\mu a^2x+a^2\lambda}{2\mu a-\lambda)}\right\}^{\frac{1}{2}}$$

When $x=a$, $t=0$ and hence $c_1=0$. When $x=b$, t gives the required value

Question 305.

(D. Jagadisan):—If the two quadrics $Ax^2+By^2+Cz^2+2\,Fyz+2\,Gzx+2\,Hxy=1$, and $ax^2+by+cz^2=1$, touch along a plane section, the plane section will pass through the centre, and will have for its equation $Lx+My+Nz=0$, where L : M : N are given by the equations

$$\frac{L^2+M^2+N^2}{A+B+C-a-b-c}=\frac{(b+c)\,L^2+(c+a)\,M^2+(a+b)\,N^2}{BC+CA+BA-F^2-G^2-H^2-(bc+ca+ab)}$$
$$=\frac{bc\,L^2+ca\,M^2+ab\,N^2}{\Delta-abc.},$$

Δ being the discriminant of the first quadric. Prove also, that the area of the plane section is

$$\pi\sqrt{\frac{A+B+C-a-b-c}{bcA+caB+abC-3abc}}$$

Solution by N. Sankara Aiyar, N. B. Pendse, and K. S. Appaswami Aiyar.

Let the plane section along which the quadrics touch be $Lx+My+Nz=k$. Then

$$Ax^2+By^2+Cz^2+2\,Fyz+2\,Gzx+2\,Hxy-1=0,$$
$$\text{and } ax^2+by^2+cz^2-1+\lambda(Lx+My+Nz-k)^2=0,$$

should be identical for some value of λ. Comparing coefficients,

$$k=0,$$
$$(a+\lambda\,L^2)/A=(b+\lambda M^2)/B=(c+\lambda N^2)/C=1,$$
$$\lambda\,MN/F=\lambda\,NL/G=\lambda LM/H=1.$$
$$\therefore\quad A+B+C-a-b-c=\lambda(L^2+M^2+N^2).$$
$$\Sigma(BC)-\Sigma(F^2)=\Sigma\,\{\,b+\lambda M^2)\,(c+\lambda N^2)-\lambda^2M^2N^2\,\}$$
$$=\Sigma(bc)+\lambda\Sigma\,\{\,L^2(b+c)\,\}\,.$$
$$\Delta=ABC+2\,FGH-\Sigma(A.F^2).\quad =abc+\lambda\Sigma(bc\,L^2).$$

Hence, the equations for L : M : N given in the question.

Again, the area of a central section(L,M,N) of $ax^2+by^2+cz^2=1$,is

$$\frac{\pi\,\sqrt{L^2+M^2+N^2}}{\sqrt{(bc\,L^2+ca\,M^2+ab\,N^2)}}=\frac{\pi\,\sqrt{(A+B+C-a-b-c)}}{\sqrt{(bc\,A+ca\,B+ab\,C)}}$$

Question 308.

(S. Ramanujan) :—Shew that—

(1) $\displaystyle\int_0^{\pi/2}\theta\cot\theta\log\sin\theta\,d\theta=-\frac{\pi^3}{48}-\frac{\pi}{4}(\log 2)^2.$

(2) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}}\frac{\sin^{-1}x}{x}\,dx-\tfrac{1}{2}\int_0^1\frac{\tan^{-1}x}{x}\,dx=\frac{\pi}{8}\log 2.$

Solution by K. J. Sanjana, M A.

We have $I_1 = \frac{1}{2}\int_0^{\pi/2} \theta \frac{d}{d\theta}(\log \sin\theta)^2 d\theta = \frac{1}{2}\Big\{\theta(\log \sin\theta)^2\Big\}_0^{\pi/2}$

$$-\frac{1}{2}\int_0^{\pi/2} (\log \sin \theta)^2 d\theta.$$

The first term vanishes: in the second term

$$\log \sin\theta = -\log 2 - \cos 2\theta - \tfrac{1}{2}\cos 4\theta - \tfrac{1}{3}\cos 6\theta \,..$$
$$= -(\log 2) - (S), \text{ suppose.}$$

Thus $I_1 = -\frac{1}{2}\int_0^{\pi/2} \{ (\log 2)^2 + \log 2.\, S + S^2 \} d\theta$

$$= -\frac{\pi}{4}(\log 2)^2 - \log 2. \int_0^{\pi/2} S d\theta - \frac{1}{2}\int_0^{\pi/2} S^2 d\theta.$$

As S contains cosines of even multiples of θ, its integral from 0 to $\pi/2$ vanishes. In S^2 the only effective part is

$$\cos^2 2\theta + \tfrac{1}{4}\cos^2 4\theta + \tfrac{1}{9}\cos^2 6\theta + \ldots$$
$$= \tfrac{1}{2}\{ 1 + \cos 4\theta + \tfrac{1}{4}(1+\cos 8\theta) + \tfrac{1}{9}(1+\cos 12\theta) \ldots \}$$
$$= \frac{1}{2.1^2} + \frac{1}{2 \cdot 2^2} + \frac{1}{2 \cdot 3^2} + \ldots + \tfrac{1}{2}(\cos 4\theta + \tfrac{1}{4}\cos 8\theta + \tfrac{1}{9}\cos 12\theta \ldots)$$

The integral of the terms in the small bracket also vanishes. Hence finally

$$I_1 = -\frac{\pi}{4}(\log 2)^2 - \tfrac{1}{2}\cdot\frac{\pi^2}{12}\cdot\frac{\pi}{2}$$

In the two parts of I_2, put $x = \sin\theta$, $x = \tan, \theta$ respectively; we get

$$I_2 = \int_0^{\pi/4} \theta\frac{\cos\theta}{\sin\theta} d\theta - \tfrac{1}{2}\int_0^{\pi/4} \frac{\theta d\theta}{\sin\theta\cos\theta}$$

$$= \int_0^{\pi/4} \frac{\theta}{\sin\theta}\left(\cos\theta - \frac{1}{2\cos\theta}\right) d\theta = \int_0^{\pi/4} \theta \cot 2\theta d\theta$$

$$= \tfrac{1}{4}\int_0^{\pi/2} \phi \cot \phi \, d\phi \text{, changing } 2\theta \text{ to } \phi.$$

$$= \tfrac{1}{4}\int_0^{\pi/2} \phi\frac{d}{d\phi}(\log \sin \phi)\, d\phi$$

$$= \tfrac{1}{4}\Big\{(\phi \log \sin\phi)\Big\}_0^{\pi/2} - \int_0^{\pi/2} \log \sin \phi d\phi$$

$$= \frac{\pi}{8}\log 2.$$

Partial solution by R. Srinivasan, M.A., S. Narayan B.A., L. T. and N. Sankara Aiyar, B.A.

$$+ \frac{\mu a^3}{(2\mu a - \lambda)^{\frac{3}{2}}}\cosh^{-1}\frac{(2\mu a - \lambda)x - \mu a^2}{a\mu a - \lambda)}$$

QUESTIONS FOR SOLUTION.

Errarum.—In Q. 318 the right side of the third identity should read

$$\tfrac{1}{2}\left\{-\frac{\pi^2}{4}\log 2+\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right\}$$

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333. (SELECTED) :—A schoolmaster went mad and amused himself by arranging the boys in his class. He turned the first boy down one place, the next two places, the next three places, and so on, until every boy's place had been altered at least once. Then he began again, and so on : till after 306 turnings down, all the boys got back to their original places. This disgusted him and he kicked one boy out. Then he was amazed to find that he had to operate 1120 times before all got back to their original places. How many boys were there in the class ?

334. (M. T. NARANIENGAR, M.A.) :—If (h, k) be the mean centre of the points of intersection of the second polar line of any point with respect to $x^3+y^3=a^3$, and (h',k') the middle point of the chord of contact of the same point with respect to the first polar conic, prove that

$$h/h'+k/k'=1.$$

335. (K. J. SANJANA, M.A) :—Prove the following identity—

$$4a^2p^2-(b^2+q^2-c^2-r^2)^2+4b^2q^2-(c^2+r^2-a^2-p^2)^2+ 4c^2r^2-(a^2+p^2-b^2-q^2)^2$$

$$=\Pi_{abc}+\Pi_{aqr}+\Pi_{brp}+\Pi_{cpq},$$

where $\Pi_{xyz}\equiv(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$; and point out its relation to the quadrilateral ABCD for which $BC=a$, $CA=b$, $AB=c$ $DA=p$, $DB=q$, $DC=r$.

336. (K. J. SANJANA, M.A.) :—Shew that

(i) $$\int_0^{\pi/2}(\log\sin\theta)^r d\theta=(-1)^r\lfloor r\left\{\frac{1}{1^{r+1}}+\frac{1}{2}\cdot\frac{1}{3^{r+1}}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^{r+1}}+\cdots\right\};$$

(ii) $$\int_0^{\pi/2}(\log\sin\theta)^r\sec\theta\, d\theta=(-1)^r\lfloor r\left\{\frac{1}{1^{r+1}}+\frac{1}{3^{r+1}}+\frac{1}{5^{r+1}}+\cdots\right\};$$

(iii) $$\int_0^{\pi/2}(\log\sin\theta)^r\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}d\theta=(-1)^r\lfloor r\left\{\frac{1}{1^{r+1}}+\frac{1}{2^{r+1}}+\frac{1}{3^{r+1}}+\cdots\right\}.$$

Also evaluate the integrals when r is odd. [$(-1)^r$ on the right will disappear on putting log cosec θ on the left side.]

337. (S. Narayanan, B.A., L.T.) :—ABC is a triangle. D and E are the points of contact with BC of the incircle and the excircle corresponding to B, and I is the incentre. If X, Y be the midpoints of AD, CE respectively, shew that IC is parallel to XY.

338. (S. Narayanan, B.A., L.T.) :—"If two pairs of opposite vertices of any complete quadrilateral be conjugate for a conic, so also is the third pair." Hesse's Theorem.

Prove this, from elementary considerations, *without* employing the methods of reciprocation or invariants.

339. (M. Bhimasena Rao,) :—The incircle of a triangle ABC touches AC, AB in E, F ; and E′, F′ are the mid-points of AC and AB. Show that the radical axis of the circles described on EF′ E′F as diameters passes through the in-Feuerbach point.

340. (M. Bhimasena Rao,) :—DEF is the pedal triangle of P with respect to a triangle ABC. If AD, BE, CF are concurrent at Q, show that PQ passes through the circumcentre of DEF and through a fixed point.

341. (Zero) :—P, Q are inverse points with respect to a circle ; any straight line through P meets the diameter perpendicular to PQ at R. If QN be the perpendicular on this line, shew that

$$RP.RN = RA^2,$$

where A is the point of intersection of PQ and the circle.

342. (A. A. Krishnaswami Ayengar) :—ABCD is a parallelogram A hyperbola is circumscribed to ABC so that one of its asymptotes passes through D. Prove geometrically that the other asymptote touches a fixed hyperbola whose asymptotes are B.A., B.C.

343. (M. Kannan, B.A., L.T.) :—If ρ_1, ρ_2, ρ_3 be the radii of curvature at the points of contact of parallel tangents to a cardioid shew that $\rho_1 \pm \rho_2 \pm \rho_3 = 0$.